1. Details of Module and its structure

Module Detail	
Subject Name	Physics
Course Name	Physics 01 (Physics Part-1, Class XI)
Module Name/Title	Unit 2, Module 7, **Projectile motion** Chapter 4, Motion in a plane
Module Id	Keph_10402_eContent
Pre-requisites	Basic Trigonometry, kinematics motion in one dimension
Objectives	**After going through this module, the learners will be able to** • Understand motion in a plane or motion in 2 dimension • Apply vectors to understand motion in 2 dimensions • Consider projectiles as special case of 2 D motion • Apply equations of motion to solve problems on projectile motion • Derive equations relating horizontal range, vertical range velocity of projection and angle of projection
Keywords	Motion in 2 dimension, resolution of vectors, projectiles motion, horizontal range, vertical range, angle of projection

2. Development Team

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1. UNIT SYLLABUS

Chapter 3: Motion in a straight line

Frame of reference, motion, position –time graph Speed and velocity
Elementary concepts of differentiation and integration for describing motion, uniform and non-uniform motion, average speed and instantaneous velocity, uniformly accelerated motion, velocity –time and position time graphs relations for uniformly accelerated motion - equations of motion (graphical method).

Chapter 4: Motion in a plane

Scalar and vector quantities, position and displacement vectors, general vectors and their notations, multiplication of vectors by a real number , addition and subtraction of vectors, relative velocity, unit vector, resolution of a vector in a plane, rectangular components ,scalar and vector product of vectors
Motion in a plane, cases of uniform velocity and uniform acceleration projectile motion uniform circular motion.

2. MODULE-WISE DISTRIBUTION OF UNIT SYLLABUS 10 Modules

The above unit is divided into 10 modules for better understanding.

Module 1	• Introduction to moving objects • Frame of reference, • limitations of our study • treating bodies as point objects
Module 2	• Motion as change of position with time • Distance travelled unit of measurement • Displacement negative, zero and positive • Difference between distance travelled and displacement • Describing motion by position time and displacement time graphs
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Module 4	• Accelerated motion • Rate of change of speed, velocity • Derivation of Equations of motion
Module 5	• Application of equations of motion • Graphical representation of motion • Numerical

Module 6	• Vectors • Vectors and physical quantities • Vector algebra • Relative velocity • Problems
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Module 8	• Circular motion • Uniform circular motion • Constant speed yet accelerating • Derivation of $a = \frac{v^2}{r}$ or $\omega^2 r$ • direction of acceleration • If the speed is not constant? • Net acceleration
Module 9	• Numerical problems on motion in two dimensions • Projectile problems
Module 10	• Differentiation and integration • Using logarithm tables

Module 7

3. WORDS YOU MUST KNOW

Let us remember the words we have been using in our study of this physics course

- Rigid body: An object for which individual particles continue to be at the same separation over a period of time.
- Point object: If the position of an object changes by distances much larger than the dimensions of the body the body may be treated as a point object.
- Frame of reference: Any reference frame the coordinates(x, y, z), which indicate the change in position of object with time.
- Inertial frame: Is a stationary frame of reference or one moving with constant speed.
- Observer: Someone who is observing objects.
- Rest: A body is said to be at rest if it does not change its position with surroundings.
- Motion: A body is said to be in motion if it changes its position with respect to its surroundings.
- Time elapsed: Time interval between any two observations of an object.
- Motion in one dimension: When the position of an object can be shown by change in any one coordinate out of the three (x, y, z), also called motion in a straight line.
- Motion in two dimension: When the position of an object can be shown by changes any two coordinate out of the three (x, y, z), also called motion in a plane.
- Motion in three dimension: When the position of an object can be shown by changes in all three coordinate out of the three (x, y, z).

- Distance travelled: The distance an object has moved from its starting position SI unit m, this can be zero, or positive.

- Displacement: The distance an object has moved from its starting position moves in a particular direction.SI unit: m, this can be zero, positive or negative.

- Path length: Actual distance is called the path length.

- Position time, distance time, displacement time graph: These graphs are used for showing at a glance the position, distance travelled or displacement versus time elapsed.

- Speed: Rate of change of distance is called speed its SI unit is m/s.

- Average speed: Total path length divided total time taken for the change in position.

- Velocity: Rate of change of position in a particular direction is called velocity, it can be zero, negative and positive, and its SI unit is m/s.

- Velocity time graph: Graph showing change in velocity with time, this graph can be obtained from position time graphs.

- Vector a physical quantity which has both magnitude and direction

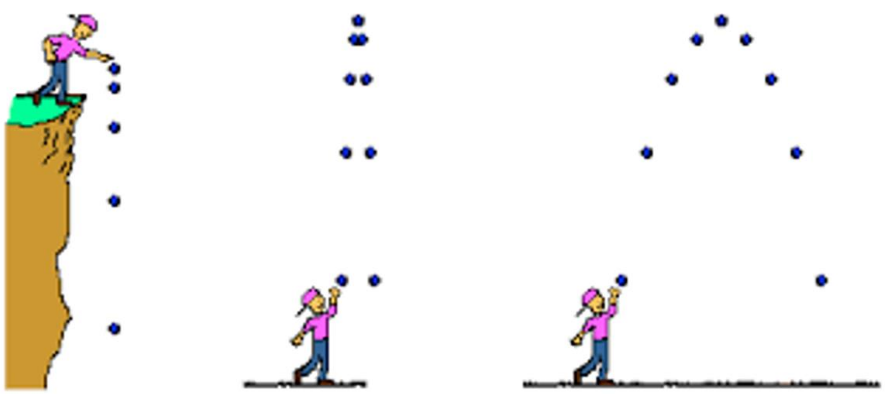
- Vector algebra mathematical rules to add subtract multiple vector quantities.

- Resolution of vectors breaking up a vector into two mutually perpendicular directions.

4. INTRODUCTION

An object that is in flight after being thrown in a particular direction or projected is called a projectile. Such projectile could be a stone, javelin, ball or any other object.

This picture shows the graphics from cricket match.



(a) (b) (c)

Notice

The ball dropped in (a),

Projected vertically upwards in (b)

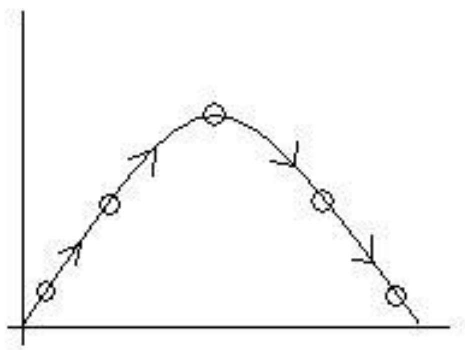
and in

(c), a ball is projected at an angle.

These are called projectile motion. In this module we will learn how to analyse this motion in 2 dimension in a simple way

5. SPECIAL WORDS

The track of a projectile is called its trajectory, it is a parabolic path in a plane or in two dimensions since the position of the object at any instant will be given by (x, y).



The motion of a projectile may be analyzed.

https://www.youtube.com/watch?v=0kSb2aWpnwM

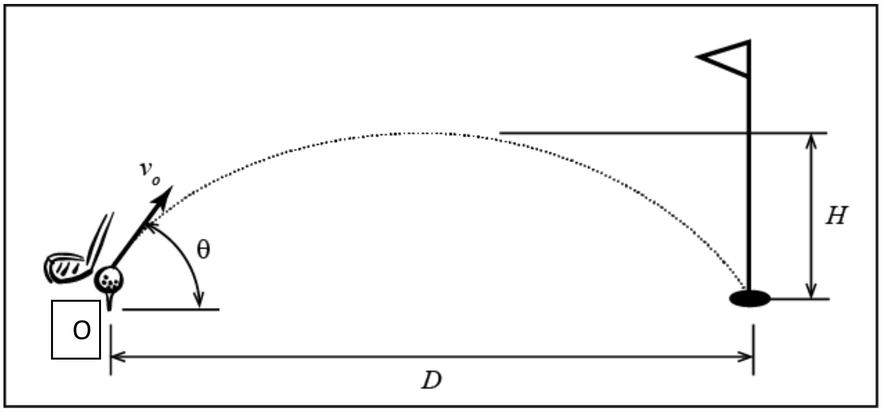
Physics classroom.com

The projectile motion may be thought of as the result of two separate, simultaneously occurring components of motions.

One component is along a horizontal direction without acceleration and **the other along the vertical direction with constant acceleration due to force of gravity.**

Interestingly it was Galileo who first stated this interdependency in his ‘Dialogue on the great world systems’ in year 1632.

Before going further we have must know the following terms:



Here,

- **Point of projection- point from where the object is projected (O).**
- **Angle of projection- angle between the directions of projection with the horizontal ϴ.**
- **Velocity of projection- initial velocity at which the object is projected v_0 or v(0)**
- **Vertical range- the maximum vertical height that the projectile attains before reaching the horizontal level containing the point of projection (H).**
- **Horizontal Range- the horizontal distance travelled by a projectile from the point of projection to a point where it touches the same horizontal level (D).**
- **Time of flight- duration for which the projectile is in its track, i.e. from point of projection to a point at the same level in the horizontal (T). Note time of flight is not the time the projectile is in the air for all conditions.**

In this module we will learn simple ways to understand motion in 2 dimensions

6. MOTION IN A PLANE:

All the above example which we discussed have a body moving in a plane, which means the position of the particle as given by (x, y) is continuously changing.

Understanding the motion of a:

Step 1

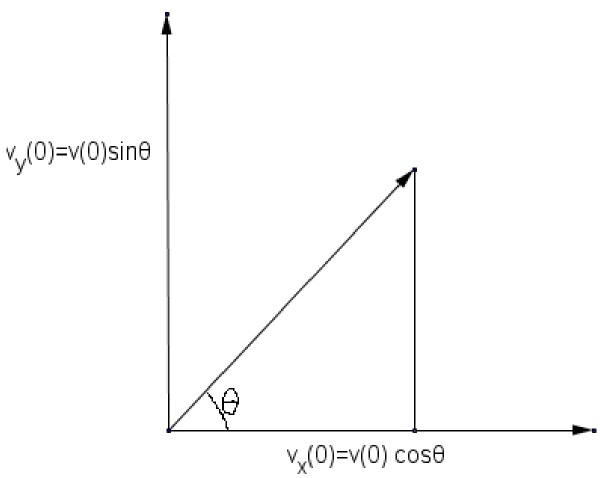
Draw a schematic diagram, mark point of projection, angle of projection, an axis x and y keeping point of projection at (0,0).

Step 2

Resolve the initial velocity vector into its components

$$\boldsymbol{along\ the\ horizontal\ v_x(0) = v(0)cos\theta}$$

$$\boldsymbol{along\ the\ vertical\ v_y(0) = v(0)sin\theta}$$



STEP 3

Apply the Kinematics Equations of motion to each component separately.

$$\boldsymbol{v(t) = v(0) + a(t-0)}$$

$$\boldsymbol{s(t) = s(0) + v(0)(t-0) - \frac{1}{2}g(t-0)^2}$$

$$\boldsymbol{v(t)^2 = v(0)^2 - 2g(s(t) - s(0))}$$

So we have two sets of kinematic equations. Why?

Because the vertical component is under the uniform acceleration while the horizontal component is in uniform motion.

Vertical direction	Horizontal direction
$\boldsymbol{initial\ velocity\ v_y(0) = v(0)sin\theta}$	$\boldsymbol{initial\ velocity\ v_x(0) = v(0)cos\theta}$
Acceleration = - g m/s^2	Acceleration = 0
Velocity at instant t $= \boldsymbol{v_y(t)}$ $\boldsymbol{v_y(t) = v_y(0) - g(t-0)}$	Velocity at instant t $= \boldsymbol{v_x(t)} =$ $\boldsymbol{v_x(0) = v(0)cos\theta}$
Vertical $\boldsymbol{distance\ travelled}$ $\boldsymbol{s_y(t) = s_0 + v_y(0)t - \frac{1}{2}gt^2}$	Horizontal distance travelled $s_x = s_0 + v_x(0)t$
Also $v_y(t)^2 = v_y(0)^2 - 2gs_y$	

Remember:

Once the projectile is in the air, there are two forces acting on it

i) **Gravity**

ii) Air resistance

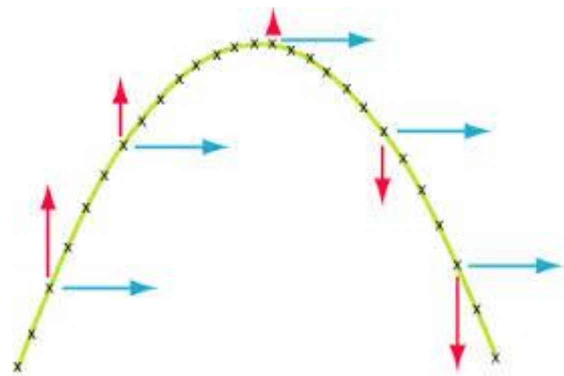
If we choose to ignore air resistance, then there is only one force -- gravity

The gravitational force causes acceleration in the vertically downward direction = g = −9.8m/s²

(value of g changes according to the location of the body),

Hence there is no acceleration in the horizontal direction.

Now imagine the velocity vectors in the horizontal direction the **blue arrows** and the vertical direction the **red arrows.**



THINK ABOUT THESE

- ✓ **What angle of projection will cause a projectile to travel the farthest?**
- ✓ **What is the relationship between launch speed and the maximum height for a projectile projected at an angle?**
- ✓ **For a point of projection at a height of ______ meters (fill-in-the-blank), what angle will cause a projectile to travel the farthest? Does the answer depend on the launch speed or is the optimum angle the same for any launch speed?**
- ✓ **A ball is dropped freely from the top of a tower and another is thrown horizontally at the same instant, which ball will strike the ground first?**

To answer similar questions, we will take up some examples

7. PROJECTILES AROUND US

EXAMPLE:

Identify projectile motion with reasons:

a) An airplane taking off a runway

b) A ball kicked by a player

c) A ball dropped from the roof of a building

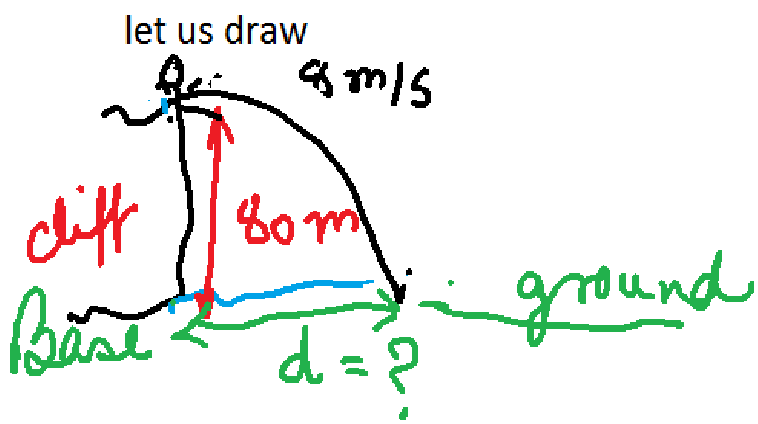
d) A ball thrown up in the air

e) Falling of rainwater from a roof spout

EXAMPLE:

A stone is thrown horizontally at speed of 8 ms^{-1} from the edge of a cliff 80 m in height. How far from the base of the cliff will the stone strike the ground?

SOLUTION



Let calculate:

The first thing to do is resolve the initial velocity which is 8ms^{-1} in the horizontal direction.

$u_x = 8ms^{-1}$ and $u_y = 0$

The vertical displacement is - 80 m.

$$-80 = 0 * t - \frac{1}{2} * 10 * t^2$$

Or

$t^2 = 16$ or $t = \pm 4s$

The stone moves a horizontal distance in the same line:

Horizontal distance = $8ms^{-1} * 4s = \mathbf{32m}$

So the stone strikes the ground at a distance of 32m from the base of the cliff.

EXAMPLE:

A toy car moves off the edge of a table, if table is 1.25m high and lands 0.4 m from the floor on which the table is placed.

1. **How long did the car take before, it hit the floor?**
2. **What was the velocity of the car when it hit the ground?**

Hint: Interpret and use equation of motion s = u t + $\frac{1}{2}$at^2

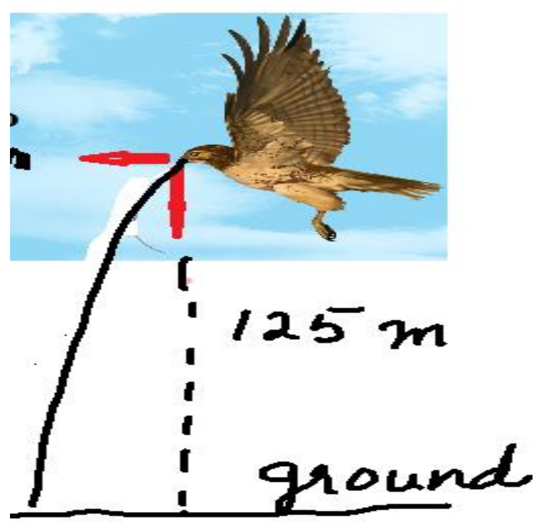
EXAMPLE:

A hawk in level flight at 125m above the ground drops a field rat it had caught. If the hawk's horizontal velocity is 20 m/s, how far ahead of the drop point will the rat land?

Why will the rat drop ahead of the drop point and not behind?

SOLUTION

The rat dead or alive was moving with the same velocity as the hawk, which is 20 ms^{-1} in the direction of the arrow.



Once it leaves the beak, it traverses a parabolic track.

The horizontal velocity is 20 ms^{-1}.

The vertical velocity at the instant of drop =0.

Let us apply the equation:

S = u t + $\frac{1}{2}$ a t^2

$$-125 = 0 - \frac{1}{2}10t^2$$

t = 5 s

Hence, the time the rat takes to reach the ground is 5 s

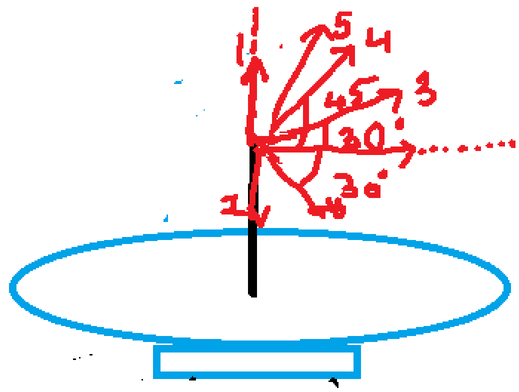
$The\ rat\ will\ strike\ the\ ground$ after 5 s at a distance

$d = v\ \times t = 20 m\, s^{-1} \times\ 5\, s = 100\ m$

i.e. 100 m away from the drop point.

EXAMPLE:

A fountain in a garden effuses water at a speed of $10 ms^{-1}$. It projects the water at vertically upwards, vertically downwards, and at angles of 30°, 45°, 60° above the horizontal and 30° below the horizontal. Draw trajectories and calculate the diameter of the circular top fountain bowl required to avoid spilling of water.



SOLUTION

The vertical and downward water jets fall vertically downwards.

So we need to study only the jets which make an angle of 30°, 45°, 60° with the horizontal.

Horizontal range has to be calculated for each angular jet

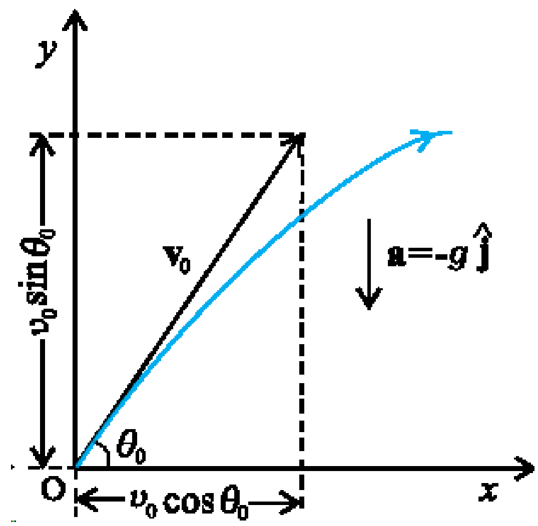
So what do we need?

i) **Horizontal component of the jet.**
ii) **Height of the fountain nozzle above the fountain bowl.**

8. DERIVATION OF EQUATIONS RELATING HORIZONTAL RANGE, VERTICAL RANGE, VELOCITY OF PROJECTION AND ANGLE OF PROJECTION

As an application of the ideas developed in the previous sections, we consider the motion of a projectile. An object that is in flight after being thrown or projected is called a projectile. Such a projectile might be a football, a cricket ball, a baseball or any other object. The motion of a projectile may be thought of as the result of two separate, simultaneously occurring components of motions. One component is along a horizontal direction without any acceleration and the other along the vertical direction with constant acceleration due to the force of gravity.

Suppose that the projectile is launched with velocity v_o that makes an angle θ_o with the x-axis as shown in fig.



After the object has been projected, the acceleration acting on it is that due to gravity which is directed vertically downward:

$a = -g\,\hat{j}$

Or, $a_x = 0$, $a_y = -g$

The components of initial velocity v_o are:

$v_{ox} = v_o \cos\theta_o$

$v_{oy} = v_o \sin\theta_o$

If we take the initial position to be the origin of the reference frame as shown in the fig in, we have:

$x_o = 0,\ y_o = 0$

We can write the equation

$$x = v_{ox}\, t = (v_o \cos\theta_o)\, t$$

and $y = (v_o \sin\theta_o)\, t - \frac{1}{2} g\, t^2$

The components of velocity at time t can be obtained

$v_x = v_{ox} = v_o \cos\theta_o$

$v_y = v_o \sin\theta_o - g\, t$

These equations give us the x-, and y-coordinates of the position of a projectile at time t in terms of two parameters — initial speed v_o and projection angle θ_o.

Notice that **the choice of mutually perpendicular x- and y-directions for the analysis of the projectile motion has resulted in simplifying the situation.**

One of the components of velocity, i.e. x-component remains constant throughout the motion and **only the y- component changes, like an object in free fall in vertical direction. This is shown graphically at few instants:**

$$\theta = \tan^{-1}\frac{v_y}{v_x}$$

Note that at the point of maximum height, $v_y = 0$.

EQUATION OF PATH OF A PROJECTILE

What is the shape of the path followed by the projectile?

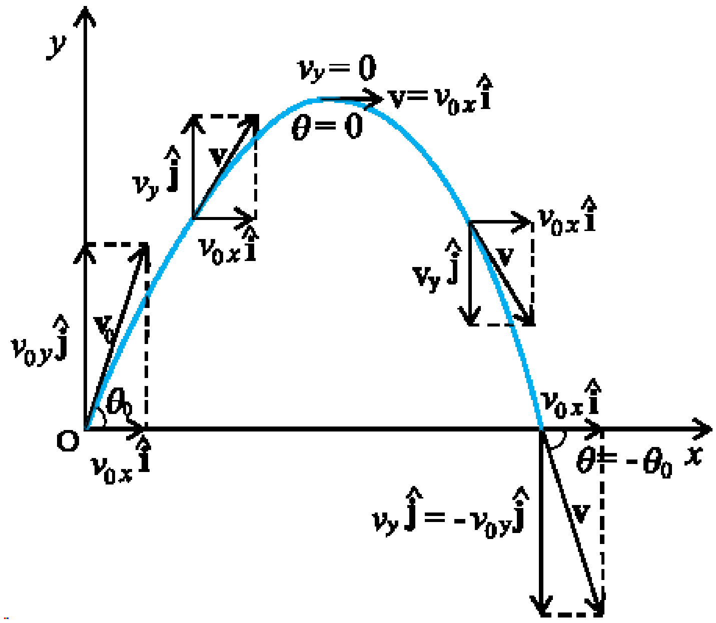
This can be seen by eliminating the time between the expressions for x and y

We obtain:

$$\mathbf{y = (\tan\theta_0)x - \frac{g}{2(v_0\cos\theta_0)^2}\,x^2}$$

Now, since g, θ_o and v_o are constants,

This equation is of the form **y = a x + b x²**, in which a and b are constants. **This is the equation of a parabola,** i.e. the path of the projectile is a parabola Fig.



TIME OF MAXIMUM HEIGHT

How much time does the projectile take to reach the maximum height?

Let this time be denoted by **t**. Since at this point, $v_y = 0$, we have from

$v_y = v_o \sin\theta_o - g\, t = 0$

or

$$\mathbf{t} = \frac{\mathbf{v_0 \sin\theta_0}}{\mathbf{g}}$$

The total time T = 2t during which the projectile is in flight

This is **known as the time of flight of the projectile.**

Time of flight, T =2t =

$$T = \frac{\mathbf{2v_0 \sin\theta_0}}{\mathbf{g}}$$

We note that T= 2 t, which is expected because of the symmetry of the parabolic path.

MAXIMUM HEIGHT OF A PROJECTILE:

The maximum height h_m reached by the projectile can be calculated by substituting $t = t_m$ in

$$y = h_m = (v_0 \sin\theta_0)\left(\frac{v_0 \sin\theta_0}{g}\right) - \frac{g}{2}\left(\frac{v_0 \sin\theta_0}{g}\right)^2$$

$$\mathbf{h_m} = \frac{(v_0 \sin\theta_0)^2}{2g}$$

HORIZONTAL RANGE OF A PROJECTILE

The horizontal distance travelled by a projectile from its initial position (x =0, y=0) to the position where it passes y = 0, again during its fall is called the horizontal range, R.

It is the distance travelled during the time of flight T.

Therefore, the range R is

$$R = (v_0 cos\theta_0)T$$

$$= \frac{(v_0\cos\theta_0)(2v_0\sin\theta_0)}{g}$$

Or, $$R = \frac{{v_0}^2 \sin 2\theta_0}{g}$$

The equation shows that for a given projection velocity v_o, R is maximum when sin $2\theta_0$ is maximum, i.e., when $\theta_0 = 45^0$.

The maximum horizontal range for this angle of projection, which is 45° with the same initial speed of projection, at any location is, therefore,

$$R_m = \frac{{v_0}^2}{g}$$

9. NUMERICAL PROBLEMS ON PROJECTILE MOTION

EXAMPLE

Galileo, in his book two new sciences, stated that “for elevations which exceed or fall short of 45° by equal amounts, the ranges are equal”. Prove this statement.

SOLUTION:

For a projectile launched with velocity v_o at an angle θ_o, the range is given by

$$R = \frac{v_0^2 \sin 2\theta_0}{g}$$

Now, for angles, (45° + α) and (45° – α), $2\theta_o$ is (90° + 2α) and (90° – 2α), respectively. The values of sin (90° + 2α) and sin (90° – 2α) are the same, equal to that of cos 2α.

Therefore, ranges are equal for elevations which exceed or fall short of 45° by equal amounts α.

EXAMPLE

A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of 15 m s^{-1}. Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed with which it hits the ground.

(Take g = 9.8 m s^{-2}).

SOLUTION:

We choose the origin of the x- and y-axis at the edge of the cliff and t = 0 s at the instant the stone is thrown.

Choose the positive direction of x-axis to be along the initial velocity and the positive direction of y-axis to be the vertically upward direction. The x-, and y- components of the motion can be treated independently.

The equations of motion are:

$\mathbf{x(t) = x_o + v_{ox}\, t}$

$\mathbf{y(t) = y_o + v_{oy}\, t + (1/2)\, a_y\, t^2}$

Here, $x_o = y_o = 0$, $v_{oy} = 0$, $a_y = -g = -9.8$ m s^{-2},

$v_{ox} = 15$ m s^{-1}.

The stone hits the ground when y(t) = – 490 m.

$$= -490m = -\frac{1}{2}(9.8)t^2$$

Notice the – sign as this distance is on the negative of our chosen frame of reference.

This gives **t =10 s.**

The velocity components are $\mathbf{v_x(t) = v_{ox}}$ and $\mathbf{v_y(t) = v_{oy} - g\ t}$

So that when the stone hits the ground:

$v_{ox} = 15\ m\ s^{-1}$

$v_{oy} = 0 - 9.8 \times 10 = -98\ m\ s^{-1}$

Therefore, the speed of the stone is

$$\sqrt{v_x{}^2 + v_y{}^2} = \sqrt{15^2 + 98^2} = \mathbf{99\ m/s}$$

Convert the speeds into km/h and enjoy the feel of the result.

Initial velocity is 15 ms^{-1} = 54 km h^{-1}

Final velocity = 99ms^{-1} = 365.4km h^{-1}

Imagine a vehicle moving at that speed!!!

EXAMPLE

A cricket ball is thrown at a speed of 28 m s^{-1} in a direction 30° above the horizontal. Calculate (a) the maximum height, (b) the time taken by the ball to return to the same level, and (c) the distance from the thrower to the point where the ball returns to the same level.

SOLUTION

(a) The maximum height is given by

$$h_m = \frac{(v_0 \sin \theta_0)^2}{2g} = \frac{(28\ \sin 30^0)^2}{2(9.8)}\ m = \frac{v_0 \sin \theta_0{}^2}{2g} = \frac{14 \times 14}{2 \times 9.8} = \mathbf{10.0\ m}$$

(b) The time taken to return to the same level is

$$T_f = \frac{2v_0 sin\theta_0}{g} = \frac{(2 \times 28 \times sin30°)}{9.8} = \frac{28}{9.8} = 2.9\ s$$

(c) The distance from the thrower to the point where the ball returns to the same level is

$$= \frac{v_0{}^2 \sin 2\theta_0}{g} = \frac{28 \times 28 \times \sin 60^0}{9.8} = \mathbf{69\ m}$$

NEGLECTING AIR RESISTANCE - WHAT DOES THE ASSUMPTION REALLY MEAN?

While treating the topic of projectile motion, we have stated that we assume that the air resistance has no effect on the motion of the projectile. You must understand what the statement really means.

Friction, force due to viscosity, air resistance are all dissipative forces.

In the presence of any of such forces opposing motion, any object will lose some part of its initial energy and consequently, momentum too. Thus, a projectile that traverses a parabolic path would certainly show deviation from its idealized trajectory in the presence of air resistance.

It will not hit the ground with the same speed with which it was projected from it. In the absence of air resistance, the x-component of the velocity remains constant and it is only the y component that undergoes a continuous change.

However, in the presence of air resistance, both of these would get affected. That would mean that the range would be less than the one given by our equation.

Maximum height attained would also be less than that predicted by the equation.

Can you then, anticipate the change in the time of flight?

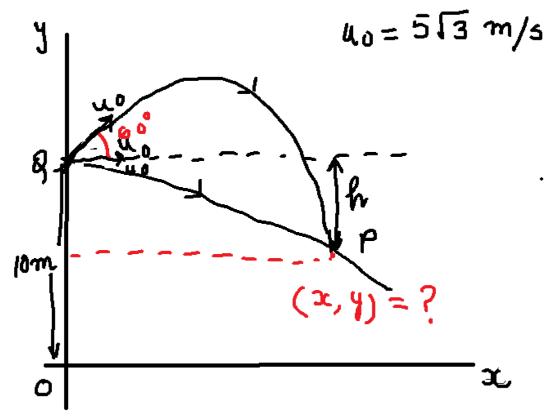
In order to avoid air resistance, we will have to perform the experiment in vacuum or under low pressure, which is not easy. This is different from practical real life situations.

Do you think a fielder in the game of cricket may miss a ‘catch because of air resistance?

When we use a phrase like ‘neglect air resistance’, we imply that the change in parameters such as range, height etc. is much smaller than their values without air resistance. The calculation without air resistance is much simpler than that with air resistance.

TRY THESE YOURSELF:

- **A ball is dropped freely from top of a bridge and another thrown horizontally at the same, which ball will hit the water below earlier?**
- **Suresh projected a stone with speed V_1= from a point o making an angle of 30^0 with the vertical, at the same instant Rita throws a ball vertically upwards with a velocity V_2 from a point A as shown. They reach a height H simultaneously. H is the highest point attained by the stone thrown by Suresh. Calculate the ratio $\frac{V_1}{V_2}$.**
- **A ball is dropped vertically from a height D above the ground. It hits the ground and bounces up vertically to a height D/2. Draw the graph between velocity and height.**
- **A batsman hits a pitched cricket ball at a height of 1.2m above the ground so that its angle of projection is 45^0**
- **Two rocket fires crackers are launched from a roof of a building 10m. Both are shot with the same speed $5\sqrt{3}$ m/s with a certain interval gap. One is fired horizontally and the other upwards at an angle of 60^0. These collide in air at a point P.**



a) **Find the time interval between the firings**

b) **Determine the** coordinates of point P

- **A projectile is launched horizontally at a speed of 30 ms^{-1} from different floors of building at a height of 20m, 30m, 40m, 50m and 60m.**
- **Draw a graph to show the relationship between height of point of projection and horizontal distance from the base of the building for each projectile?**
- **A ball is kicked with an initial velocity of $2\sqrt{hg}$; it just clears two walls of equal height h which are at a distance of 2h from each other. Calculate the duration for which a boy standing between the walls will see the ball?**

10. SUMMARY

You have learnt:

- Motion in two dimensions
- Graphical and mathematical methods to describe motion in two dimensions.
- Projectile and way to describe projectile motion by taking the components of motion along x and y directions.
- One component along the x axis remained in uniform motion.
- The vertical component along y-axis is accelerated motion.
- Derivation of equation of trajectory of a projectile.
- Equation relating initial velocity of projection, angle of projection, horizontal and vertical range.
- Methods of solving problems in two-dimension.